For the following problems, let $\Sigma = \{c, s\}$.

1. Prove the following language is *not* regular.

   $B_1 = \{ w \mid w \text{ contains an equal number of } c \text{'s and } s \text{'s} \}$

   - Assume $B_1$ is regular. This means the pumping lemma for regular languages holds – that is, *any* string $s$ of length at least $p$ that is in $B_1$ can be “pumped”.

   - Consider $s = c^p s^p$. Note $s \in B_1$ and $s$ has length at least $p$, so $s$ must be able to be “pumped”.

   - Given that the part we are pumping ($y$) must be non-empty, and that it must occur in the first $p$ symbols of $s$, we know that $y$ must: consist of 1 or more $c$’s

   - Consider the string $s'$ which is created from “pumping” $y$ 0 times.

     $s' = x y^0 z = c^{p - |y|} s^p$

   - Note that $s' \notin B_1$, as our inferences about $y$ we made above mean: $p - |y| \neq p$, so there are fewer $c$’s than $s$’s

   - We therefore have a contradiction – $B_1$ is regular so $s$ must be “pumpable”, but we have shown it is not “pumpable”. We reached this contradiction by assuming $B_1$ was regular. Therefore, $B_1$ is *not* regular.

2. Prove the following language is *not* regular.

   $B_2 = \{ c^n s^m \mid n \leq m \}$

   - Consider $s = c^p s^p$

   - We know that $y$ must: consist of 1 or more $c$’s

   - $s' = x y^2 z = c^{p + |y|} s^p$

   - $s' \notin B_2$ because: $p + |y| > p$, so we have more $c$’s than $s$’s
3. Prove the following language is not regular.

\[ B_3 = \{ w \mid \text{the length of } w \text{ is a power of } 2 \} \]

- Consider \( s = c^{2p} \)

- We know that \( y \) must: consist of at least 1 and at most \( p \) c’s

- \( s' = xy^2z = c^{2^p + |y|} \)

- \( s' \notin B_3 \) because:
  - Consider the next power of 2 and how it relates to \( s' \):
  - \( 2^p < 2^p + |y| < 2^{(p+1)} \). \( |s'| \) is not a power of 2.

4. Prove the following language is not regular.

\[ B_4 = \{ wc^n | w \text{ is a string over } \Sigma \text{ of length } n, n \geq 0 \} \]

- Consider \( s = s^p c^p \)

- We know that \( y \) must: consist of 1 or more \( s \)’s

- \( s' = xy^2z = s^{p+|y|} c^p \)

- \( s' \notin B_4 \) because:
  - \( p + |y| \neq p \), so the length of the c’s “part” is
  - less than the length of the w “part”.


5. Prove the following language is *not* regular.

\[ B_5 = \{ w \mid w \text{ is a “balanced” string, with } c \text{ “opening” and } s \text{ “closing” } \} \]

(I’ve obviously made these terms up, but I’ll explain \( B_5 \) in class)

• Consider \( s = \underline{c^p s^p} \).

• We know that \( y \) must:
  _consist of 1 or more \( c \)’s_

• \( s' = \underline{xy^0z = c^{p-|y|}s^p} \)

• \( s' \notin B_5 \) because:
  _\( p - |y| < p \), so fewer “opening” \( c \)’s than closing \( s \)’s_