Objective: To experiment with recursion, and see the possible pitfalls with divide-and-conquer implementation and possible solutions using an iterative dynamic programming implementation.

We've seen several recursive "divide-and-conquer" algorithms: binary search, merge sort, quick sort. The general idea of divide-and-conquer algorithms is:

- dividing the original problem it into small problem(s) (e.g., merge sort - split into left half and right half)
- solving the smaller problem(s) (e.g., merge sort recursively solved each half)
- combine the solution(s) to smaller problem(s) to solve the original problem (e.g., merge sort - merged the two half back together to solve the original sorting problem)

Part A: Mathematics has several simple recursively defined functions. For example, the *factorial* function can be recursive defined as:

$n! = n * (n - 1)!$ for $n \geq 1$, and
$0! = 1$

Implement a recursive factorial(n) function using this recursive definition and test it with several small examples (e.g., 3! = 3*2! = 3*2*1! = 3*2*1*0! = 3*2*1*1 = 6, and 5! = 5*4*3*2*1*1 = 120). One problem with using the above formula **in most languages** is that n! grows very fast and overflows an integer representation. For example, 52! = 80,658,175,170,943,878,571,660,636,856,403,766,975,289,505,440,883,277,824,000,000,000,000 which is much, much bigger than can fit into a 64-bit integer representation. Fortunately, Python does not suffer from this problem, since it automatically switches to unlimited Long integers for really big values like this. Use your function to calculate 52!.

Another simple recursive definition is the *Fibonacci sequence*: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... Where each number in the sequence is the sum of the previous two numbers. A recursive definition to calculate the n^{th} number in the Fibonacci sequence is: (Notice that we start numbering the position in the Fibonacci sequence from 0)

Fib(n) = Fib(n-1) + Fib(n-2) for n > 1, and
Fib(1) = 1
Fib(0) = 0

a) Implement a recursive function Fib(n) to calculate the n^{th} number in the Fibonacci sequence. Test is with small values like Fib(8) which is 21, and Fib(10) which is 55.

Use a simple main function as shown below to print the first 50 values of the fibonacci sequence.

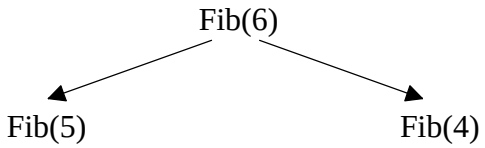
```
def main():
  for n in range(50):
     print "The", n, "th fibonacci value is", Fib(n)
```

b) What do you observe about the time it takes for each Fib( ) call to complete?

c) Why do you thing that the recursive fibonacci is so slow?

After you have implemented factorial and fibonacci AND answered the above questions, raise your hand and explain your answers.

Part B. Complete the following recursion "calling tree" similar to what we did in class yesterday for quickSort.



a) Approximately how much more work is performed by calculating Fib(n+1) than Fib(n) for some large n?

b) What would you guess would be the "big-oh" notation for your recursive Fib(n) function? *O*( )

c) Much of the slowness of your "divide-and-conquer" Fibonacci function, Fib(n), is due to redundant calculations performed due to the recursive calls. How else might we calculate Fibonacci without constantly recalculating the result of smaller problem instances?

After you have tried to answered the above questions, raise your hand and explain your answers.

Part C. What you hopefully figured out from Part B is a VERY POWERFUL concept in Computer Science -- *dynamic programming*. Dynamic programming solutions eliminate the redundancy of divide-and-conquer algorithms by calculating the solutions to smaller problems first, storing their answers, and looking up their answers if later needed instead of recalculating them. In Python we could use a list to store the answers to smaller problems of the Fibonacci sequence (in most programming languages you would use an array).

To use dynamic programming when you have a recursive statement of the problem. You can do the following steps:

1) Store the solution to smallest problems, i.e., the base cases
2) Loop (no recursion needed) from the base cases up to the biggest problem of interest. On each iteration of the loop we:
   - solve the next bigger problem
   - store its result for later use so we never have to recalculate it

We'll reimplement Fib(n) that calculates the n^{th} number in the Fibonacci sequence use dynamic programming. Recall the recursive definition of the Fibonacci sequence:

Fib(n) = Fib(n-1) + Fib(n-2) for n > 1, and
Fib(1) = 1
Fib(0) = 0

a) We'll use a list called, fibonacci, to store the solution to the "smaller" problems. Complete the dynamic programming code:

```
# Dynamic programming solution to determine the nth number in the
# Fibonacci sequence.
def Fib(n):
    # Step 1:  Store base case solutions
    fibonacci = [

    # Step 2:  Loop from base case to biggest problem of interest
    for next in range(



    # return nth number in the Fibonacci sequence
    return
```

b) Rerun the simple main function to print the first 50 values of the fibonacci sequence using your dynamic programming solution.

```
def main():
  for n in range(50):
      print "The", n, "th fibonacci value is", Fib(n)
```

Note any "Wow" comments here:

c) One tradeoff of simple dynamic programming implementations is that they can require more memory since we store solutions to **all** smaller problems. Often, we can reduce the amount of storage needed if the next larger problem (and all the larger problems) don't really need the solution to the really small problems, but just the larger of the smaller problems. In fibonacci for example, when calculating the next value in the sequence we only need the previous two solution. Reimplement the Fib(n) function a third time to eliminate the list fibonacci and replace it with three variables.

After you have implemented both dynamic programming versions of fibonacci, raise your hand and explain your code.

Part D. Sometimes a solution to the ("largest") problem of interest does not require the solution to all of the smaller problems. We'd like to calculate only the solutions to smaller problems that we actually need for the problem of interest. One problem is predicting which smaller problems will be needed. One solution to this is *memoization* where you write a recursive solution like divide-and-conquer because the recursive solution starts with the largest problem and only calls smaller problems that are actually needed to solve it. To avoid the divide-and-conquer problem of recalculating smaller problems multiple times, we'll check if we previously calculated a solution to a smaller problem before doing the recursive call.

The general steps for using memoization are:
1) initialize a list (/array) of **all** smaller problem solutions to some dumby value like -1 to indicate that these problems have not yet been solved.
2) in this list set the base case solutions
3) write a recursive version like divide-and-conquer, except before doing the recursive call(s) check if your list already has an answer. If it does, use the list answer instead.

```
def fib(n):
    # Step 1 and 2:  initialize solution list and set base cases
    fibonacci = []
    fibonacci.append(0)
    fibonacci.append(1)
    for i in range(2,n+1):
        fibonacci.append(-1)

    # Step 3:
    fib_helper(n, fibonacci)

    return fibonacci[n]

def fib_helper(n, fibonacci):
```

a) Write fib_helper recursive function.

After you have implemented your memoization dynamic programming version of fibonacci, raise your hand and explain your code.