Terminology:

<u>problem</u> - question we seek an answer for, e.g., "what is the largest item in an array?"

<u>instance</u> - the specific input to the problem

	0	1	2	3	4	5	6
array S:	5	10	2	15	20	1	11

largest: ?

n: 7
(number of elements)

<u>algorithm</u> - step-by-step procedure for producing a solution

<u>basic operation</u> - fundamental operation in the algorithm (i.e., operation done the most)

<u>problem size</u> - input size. For algorithms involving arrays, the problem size is often the number of elements.

<u>Our Goal</u>: We want to derive a function for the number of times that the *basic operation* is performed related to the *problem size.*

Example *sum* - Add all the numbers in the array *S* of *n* numbers.
Inputs: positive integer *n*, array of numbers *S* indexed from 1 to *n*.
Outputs: *sum*, the sum of the numbers in *S*.

```
number sum ( int n, const keytype S[ ] )
{
    index i;
    number result;

    result = 0;
    for (i = 1;  i <= n;  i++)
        result = result + S [ i ];
    return result;
}
```

(Note the text uses bold to indicate an array, e.g., **a**, and subscripts to indicate an array element, e.g., a_3 or a_j. However, I'll usually use square brackets to denote an array element a[3].)

For algorithm *sum*, what is the basic operation?

What is the problem size?

Run-time Analysis of *sum*

```
number sum ( int n, const keytype S[ ] )
{
    index i;
    number result;

    result = 0;                          ← Done once.  Assume time to do once is c1 = 100
    for (i = 1;  i <= n;  i++)
        result = result + S [ i ];       ] Done n times.  Assume time to do loop once is c2 = 10

    return result;
}
```

Every-case time complexity analysis, $T(n) = c_1 + c_2\, n = 100 + 10\, n$

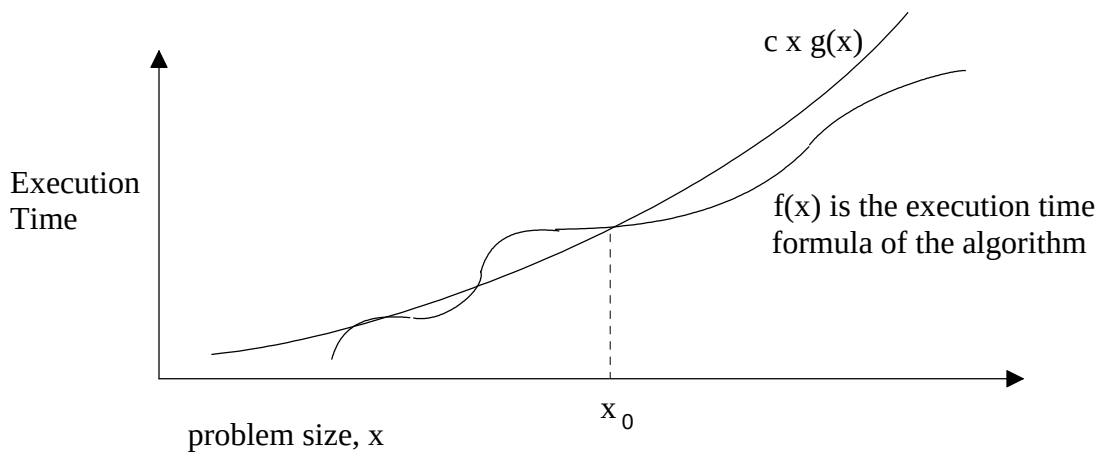
What determines the length of c_1 and c_2?

When n is "sufficiently large," the 10 n term dominates. For what value of n, does 10 n = 100?

Big-oh Definition - asymptotic upper bound

Definition 2.1: A function $f(x)$ is "Big-oh of $g(x)$" or "$f(x)$ is $O(g(x))$" or "$f(x) = O(g(x))$" if there are positive, real constants c and x_0 such that for all values of $x \geq x_0$,

$$f(x) \leq c \times g(x).$$



$T(n) = c_1 + c_2\, n = 100 + 10\, n$ is $O(\, n\,)$.

"Proof": Pick $c = 110$ and $x_0 = 1$.
Then $100 + 10\, n \leq 110\, n$ for all $n \geq 1$ since

$$100 + 10\, n \leq 110\, n$$
$$100 \leq 100\, n$$
$$1 \leq n \text{ which true for all } n \geq 1.$$

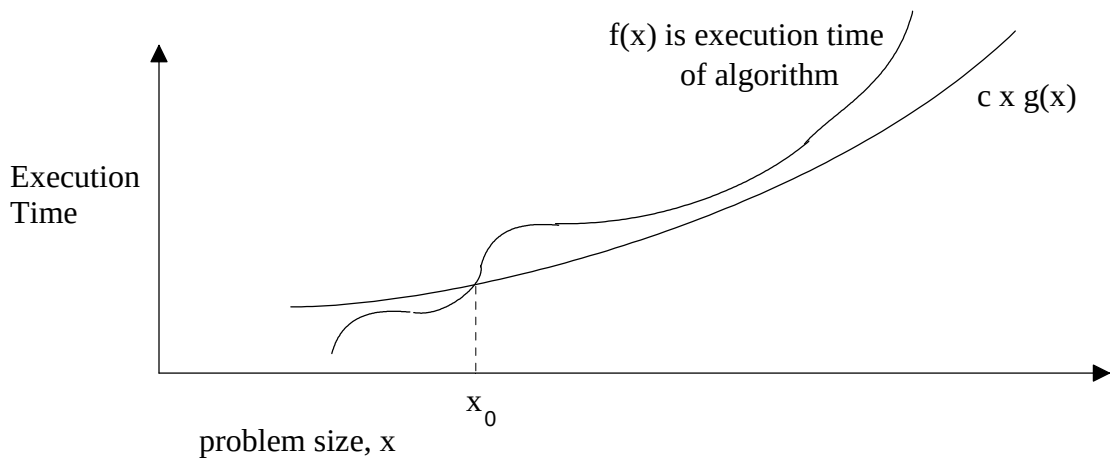
Problem with big-oh:

If T(n) is O(n), then it is also $O(n^2)$, $O(n^3)$, $O(n^3)$, $O(2^n)$, ... since these are also upper bounds.

Omega Definition - asymptotic lower bound

Definition 2.2: A function $f(x)$ is "omega of $g(x)$" or "$f(x)$ is $\Omega(g(x))$" or "$f(x) = \Omega(g(x))$" if there are positive, real constants c and x_0 such that for all values of $x \geq x_0$,

$$f(x) \geq c \times g(x).$$



$T(n) = c_1 + c_2\, n = 100 + 10\, n$ is $\Omega(\, n\,)$.

"Proof": We need to find a c and N so that the definition is satisfied, i.e.,
$100 + 10\, n \geq c\, n$ for all $n \geq N$.

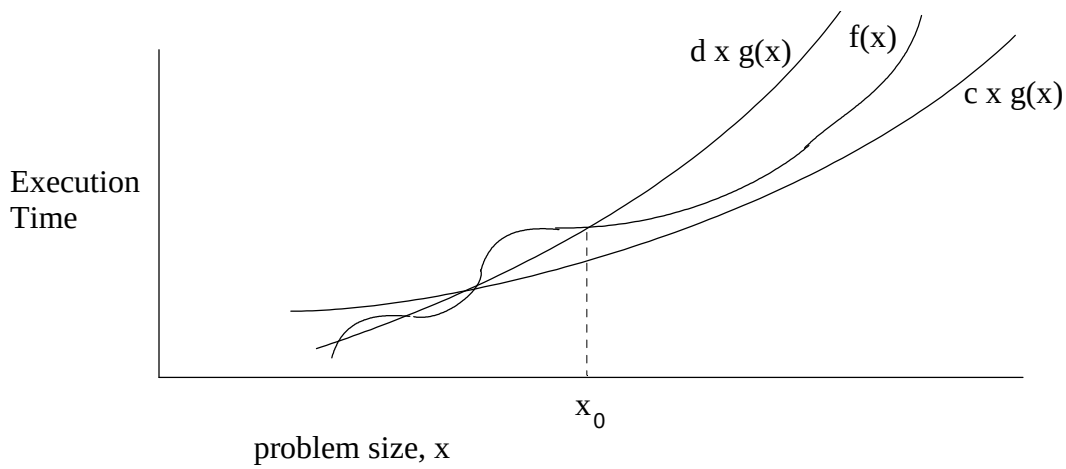
What c and x_0 will work?

Theta Definition - asymptotic upper **and** lower bound, i.e., a "tight" bound, i.e., "best" big-oh

Definition 2.3: A function $f(x)$ is "theta of $g(x)$" or "$f(x)$ is $\theta(g(x))$" or "$f(x) = \theta(g(x))$" if $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$.

Alternate Definition 2.3: A function $f(x)$ is "theta of $g(x)$" or "$f(x)$ is $\theta(g(x))$" or "$f(x) = \theta(g(x))$" if there are positive, real constants c, d, and x_0 such that for all values of $x \geq x_0$,

$$c \times g(x) \leq f(x) \leq d \times g(x).$$



$T(n) = c_1 + c_2\, n = 100 + 10\, n$ is $\theta(\, n\,)$.

Tractable versus Intractable Problems

A problem is *intractable* if it is impossible to solve it with *any* polynomial-time algorithm.

Three general categories of problems:

1. Problems with known polynomial-time algorithms, e.g., sorting an array
2. Problems that have been proven to be intractable, e.g., listing every subset of an n-element set.
3. Problems that have not proven to be intractable, but for which no known polynomial-time algorithm exists, e.g., traveling salesperson problem, TSP

Traveling Salesperson Problem, TSP - find the shortest path through a set of cities, visiting each city only one time.

Input: A map of cities including the roads between the cities, and distances along the roads. (i.e., a graph problem)
Output: A sequence of roads that will take a salesperson through every city on the map and return them to the starting city, such that they will visit each city exactly once, and will travel the shortest total distance.

The TSP is in a special class of algorithms called *NP-complete* problems. *NP-complete* problems have a critical property that if a polynomial-time algorithm can be found for one *NP-complete* problem, then all *NP-complete* problems can be solved in polynomial-time by "minor" modifications of the same algorithm.

Many bioinformatics problems are *NP-complete* problems (and thousands of non-bioinformatics problems). Since thousands of Computer Scientists have tried for about 50 years to find a polynomial-time algorithm to any of these *NP-complete* problems, it is unlikely that an exist.

NOTE: This is one of the biggest open questions in Computer Science!!!