Theorem B.1 Let the homogeneous linear recurrence equation with constant coefficients

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \ldots + a_k t_{n-k} = 0$$

be given. If the characteristic equation

$$a_0 r^k + a_1 r^{k-1} + a_2 r^{k-2} + \ldots + a_k r^0 = 0$$

has ***k distinct*** solutions $r_1, r_2, \ldots, r_k$, then the only solutions to the recurrence are

$$t_n = c_1 r_1^n + c_2 r_2^n + c_3 r_3^n + \ldots + c_k r_k^n$$

where the c_i terms are arbitrary constants. (The value of these c_i can be determined by the initial/base-case conditions, but we generally can stop here to find the theta-notation.)

Theorem B.2 Let r be a root of multiplicity m of the characteristic equation for a homogeneous linear recurrence equation with constant coefficients. Then, the general solution includes the terms

$$t_n = \ldots + c_1 r^n + c_2 n r^n + c_3 n^2 r^n + \ldots + c_{m-1} n^{m-1} r^n + \ldots$$

where the c_i terms are arbitrary constants.

1) For the recurrence:

$t_n = 7\, t_{n-1} - 15\, t_{n-2} + 9\, t_{n-3}$ for $n > 2$

$t_0 = 0$

$t_1 = 1$

$t_2 = 2$

a) Obtain the characteristic equation

b) Factor the characteristic equation to find the roots

c) Write the general solution to the characteristic equation and determine its theta notation.

Theorem B.3 A nonhomogeneous linear recurrence equation of the form

$$a_0 t_n + a_1 t_{n-1} + a_2 t_{n-2} + \ldots + a_k t_{n-k} = b^n p(n)$$

can be transformed into a homogeneous linear recurrence that has the characteristic equation

$$(a_0 r^k + a_1 r^{k-1} + a_2 r^{k-2} + \ldots + a_k r^0)(r-b)^{d+1} = 0,$$

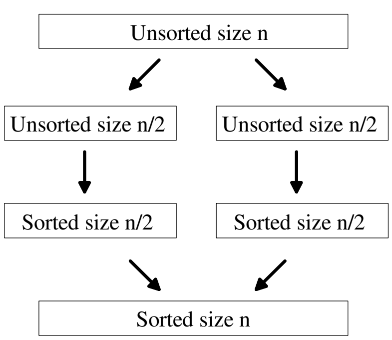
where b is a constant and $p(n)$ is a polynomial of degree d. If there is more than one term like $b^n p(n)$ on the right-hand side, each one contributes a $(r-b)^{d+1}$ term to the characteristic equation.

2) For the recurrence:
$t_n = 2\ t_{n-1} + 2^n - 1$ for $n > 0$
$t_0 = 0$
a) Obtain the characteristic equation

b) Factor the characteristic equation to find the roots

c) Write the general solution to the characteristic equation and determine its theta notation.

3) When analyzing divide-and-conquer algorithms, we often need to perform a "domain transformation" in order to apply Theorem B.3. Recall the recurrence for merge sort:



$W(n) = 2\ W(n/2) + n - 1$
$W(1) = 0$

If we assume that n is a power of 2, then $n = 2^k$ and $\log_2 n = k$.

a) What is the recurrence if we substitute 2^k for n?

b) To apply Theorem B.3 we want a linear recurrence equation ($t_n = c_1\ t_{n-1} + c_2\ t_{n-2} + \ldots$). What should we substitute for in part (a)'s answer to accomplish this?