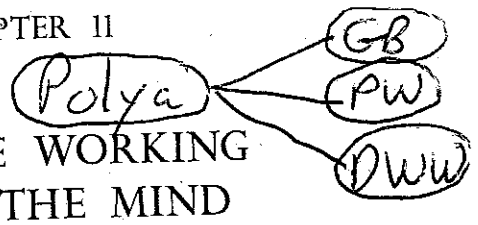
CHAPTER 11



THE WORKING OF THE MIND

Mariotte says that the human mind is like a bag: when you are thinking you are shaking the bag until something falls out of it. Hence there is no doubt that the result of thinking depends to some extent on chance. I would add that the human mind is more like a sieve: when you are thinking you are shaking the sieve until some minute things pass it. When they pass, the spying attention catches whatever seems relevant. Again, it is something like this: to catch a thief, the commander of a city orders the whole population to pass a certain gate where the man who was robbed is watching. Yet, to save time and trouble, some method of exclusion may be used. If the man robbed says that the thief was a man, not a woman, and an adult, not a youngster or a child, those not concerned are excused from passing the gate.

LEIBNITZ: *Opuscules et fragments*, p. 170.

11.1. How we think

A problem solver must know his mind and an athlete must know his body in about the same way as a jockey knows his horses. I imagine that a jockey studies horses not for the sake of pure science but to make them perform better, and that he studies more the habits and whims of individual horses than horse physiology or horse psychology in general.

What you start reading now is not a chapter in a textbook of psychology; it is not exactly a conversation between problem solvers who talk about the habits of their minds as jockeys may talk about the habits of their horses; it is, however, more like a conversation than a formal presentation.

11.2. Having a problem

An essential ingredient of the problem is the desire, the will, and the resolution to solve it. A problem that you are supposed to do and which you have quite well understood, is not yet your problem. It becomes your problem, you really have it, when you decide to do it, when you desire to solve it.

You may be involved more or less deeply in your problem—your desire to solve it may be more or less strong. Unless you have a very strong desire, your chances to solve a really hard problem are negligible.

The desire to solve your problem is a *productive* desire: it may eventually produce the solution, it certainly produces a change in your mental behavior.

11.3. Relevancy

You may have a problem so badly that the problem has you; you cannot get rid of your problem, it follows you everywhere.

A man with a problem may be obsessed by his problem. He appears absentminded; he does not notice things which appear obvious to his neighbors, and he forgets things which none of his neighbors would forget. Newton, working intensely on his problems, often forgot to eat his meals.

Yes, the problem solver's attention is *selective:* it refuses to dwell on things which appear irrelevant to his problem and espies the most minute things that appear relevant. It is a "spying" attention as Leibnitz put it.

11.4. Proximity

A student takes a written examination in mathematics. He is not required to do all the proposed problems, but he should do as many as possible. In this situation his best strategy may be to start by looking through all the problems at an appropriate pace and choose those he is most likely to master.

Observe that this supposes that the problem solver is able to assess to some extent the difficulty of his problems, that he can estimate to a degree his "psychological distance" from his problem's solution. In fact, anybody seriously concerned with his problem has a vivid feeling for the proximity of the solution and for the pace of his progress toward the solution. He may not use words but he feels keenly: "It goes well, the solution may be just around the corner," or "It goes so slowly and the solution is still far off," or "I got stuck, there is no progress at all," or "I am drifting away from the solution."

11.5. Prevision

As soon as we are seriously concerned with our problem, we try to foresee, we try to guess; we expect something, we anticipate an outline of the shape of the solution. This outline may be more or less definite—and of course it may be more or less wrong, although I would say not often very wrong.

All problem solvers guess, but the sophisticated and the unsophisticated guess somewhat differently.

A primitive person just sits there with his problem, scratching his head or chewing his pencil, waiting for a bright idea, and doing little or nothing to bring that bright idea nearer. And when the desired idea eventually appears and brings a plausible guess, he simply accepts that guess, regarding it as the solution with little or no criticism.

A more sophisticated problem solver takes his guesses more skeptically. His first guess may be: "There are 25" or "I should tell him this and that." Yet then he checks his guess and may change it: "No, not 25. Yet let me try 30" or "No. It is no use to tell him that, because he could answer thus and so. Yet I could tell him that . . . ." And eventually, by "trial and error," by successive approximations, the problem solver may arrive at the right answer, at an appropriate plan.[1]

A still more sophisticated and more experienced problem solver, when he does not succeed in guessing the whole answer, tries to guess some part of the answer, some feature of the solution, some approach to the solution, or some feature of an approach to the solution. Then he seeks to expand his guess, but also seeks opportunities to check his guess, and so he seeks to adapt his guess to the best information he can get at the moment.

Of course, both the sophisticated and the unsophisticated would like to have a really good guess, a bright idea.

And everybody would like to know what chances his guess has to come true. Such chances cannot be precisely evaluated (this is not the place to discuss remote possibilities of evaluation). Many times, however, the problem solver has a definite feeling about the prospects of his guess. Primitive people who do not even know what a proof is may have the strongest feelings about their guesses; sophisticated people may distinguish fine shades of feeling; but anybody who has conceived a guess has some feeling about the likely fate of his guess. And so we notice still another sort of feeling, besides the feelings of relevancy and proximity, in the problem solver's mind.

Is this point relevant? How far off is the solution? How good is this guess? Such questions accompany each move of the problem

[1] Sect. 2.2(1) and 2.2(5).

solver; they are more felt than formulated and the answers, too, are more felt than formulated. Do such feelings guide the problem solver or do they merely accompany his decisions? Are they causes or symptoms? I don't know, but I do know that if you do not have such feelings, you are not really concerned with your problem.

11.6. Region of search

I seldom part with my wrist watch, but when I do I usually have some trouble to find it. When I miss my watch, I habitually start looking for it at some well-defined place: on my desk, or on a certain shelf where I am used to store little belongings, or at any third place if I happen to remember that I took off my wrist watch just there.

Such behavior is typical. As soon as we are seriously concerned with our problem, we anticipate an outline of its solution. This outline may be vague, it may be hardly conscious, but it manifests itself in our behavior. We may try various solutions, but they are all alike; they are all within that preconceived, but perhaps not consciously preconceived, outline. When none of the solutions tried fits the problem, we feel lost, nothing else comes to mind; we cannot step outside that preconceived outline. We do not look for just any kind of solution, but for a certain kind, a kind within a limited outline. We do not look for a solution just anywhere in the world, but for a solution within a certain limited *region of search.*[2]

To begin our search within a likely limited region may be reasonable. When I am trying to find my missing wrist watch, it is quite reasonable not to look for it anywhere in the universe, or anywhere in the city, or anywhere in the house, but just on my desk where I found it several times in the past. It is quite reasonable to begin by seeking the unknown within that limited region, but it is unreasonable to persevere in seeking it there even when it becomes more and more clear that it is not there.

11.7. Decisions

Problem solving may be contemplative; with primitive people, it may be inarticulate brooding. Or it may be a long, strenuous, winding road to the solution, each turning of which is marked by a decision. Such decisions are prompted (or perhaps merely accompanied) by feelings of relevancy and proximity, by swelling or fading hope. Decisions and prompting feelings are seldom expressed in words, but may be occasionally:

[2] Karl Duncker, On Problem Solving, *Psychological Monographs*, vol. 58, No. 5 (1945). See p. 75.

"Now, let me look at this."

"No, there is not much to see here. Let me look at that."

"There is not much to see here either, but there is something in the air. Let me look at it a little longer."

An important type of decision is to enlarge the region of search, to discard a limitation the narrowness of which starts giving us an oppressive feeling.

11.8. Mobilization and organization

The problem solver's mental activity is very imperfectly known and its complexity may be unfathomable. Yet one result of this activity is perfectly obvious: as the problem solver advances, he collects more and more material.

Let us compare the problem solver's conception of a mathematical problem at the beginning and at the end of his work. When the problem arises, there is a simple picture: the problem solver sees his problem as an undivided whole without details, or with very few details; for instance, he may see just the principal parts, unknown, data, and condition, or hypothesis and conclusion. Yet the final picture is very different: it is complex, full of added details and materials the relevancy of which the problem solver could hardly have suspected at the outset. There are auxiliary lines in the originally almost empty geometric figure, there are auxiliary unknowns, there are materials from the formerly acquired knowledge of the problem solver, especially theorems applied to the problem. That just these theorems will be applicable, the problem solver did not foresee at all at the beginning.

Where do all these materials, auxiliary elements, theorems, etc. come from? The problem solver has collected them; he had to extract them from his memory and purposefully connect them with his problem. We call such collecting *mobilization* and such connecting *organization*.[3]

Solving a problem is similar to building a house. We must collect the right material, but collecting the material is not enough; a heap of stones is not yet a house. To construct the house or the solution, we must put together the parts and organize them to a purposeful whole.

Mobilization and organization cannot actually be separated; they are complementary aspects of the same complex process—of our work aimed at the solution. Such work, when intensive, brings into play all our psychical resources, requires the whole gamut of our mental activities, and presents an inexhaustible variety of aspects. We may be tempted to distinguish some of the manifold mental operations involved and describe them by such terms as isolation and combination, recognizing and remembering, regrouping and supplementing.

[3] *Cf.* ex. 2.74.

appropriate auxiliary line. Yet sometimes we can take the decisive step without introducing any new line, just by conceiving the lines already present in a new fashion. For example, we may notice that certain lines form a pair of similar triangles. In noticing this familiar configuration, we recognize hitherto unobserved relations between the elements of the figure, we see the elements differently grouped, we see a new structure, we see the figure as a better arranged, more harmonious, more promising whole—we have restructured the problem material.

Regrouping may involve a change in emphasis. Elements and relations which were in the foreground before the regroupment may now surrender their privileged place and recede into the background; they may even recede so far that they practically drop out from the conception of the problem. For better organization we must now and then reject things which we thought relevant some time ago. Yet, on the whole, we add more than we reject.

11.11. Isolation and combination

When we are examining a complex whole, our attention may be attracted now by this detail and then by another. We concentrate on a certain detail, we focus on it, we emphasize it, we single it out, we distinguish it from its surroundings, in one word, we *isolate* it. Then the spotlight shifts to another detail, we isolate still another detail, and so on.

After examining various details and revaluing some of them, we may feel the need of visualizing again the situation as a whole. In fact, after the revaluation of some details, the appearance of the whole, the "vue d'ensemble," the "Gestalt" may have changed. The combined effect of our reassessment of certain details may result in a new mental picture of the whole situation, in a new, more harmonious *combination* of all the details.

Isolation and combination may advance the solution in complementing each other. Isolation leads to decomposing the whole into its parts, a subsequent combination reassembles the parts into a more or less different whole. Decomposed and recombined, again decomposed and again recombined, our view of the problem may evolve toward a more promising picture.

11.12. A diagram

A diagrammatic summary of the foregoing sections is offered by Fig. 11.1, which the reader should take for what it is worth. Nine terms are arranged in a square; one occupies the center of the square, four

The following lines attempt to describe these activities. Of course, the reader should not expect, and could not reasonably expect, hard and fast distinctions or rigid and exhaustive definitions.

11.9. Recognizing and remembering

In examining our problem we cheer up when we *recognize* some familiar feature. Thus, in examining a geometric figure, we may recognize with pleasure a triangle not noticed before, or a pair of similar triangles, or some other intimately known configuration. Examining an algebraic formula we may recognize a complete square, or some other familiar combination. Of course, we may also recognize, and it may be very useful to recognize, some more complex situation to which we cannot yet attach a name and for which we have not yet a formal definition, but which strikes us as familiar and important.

We have good reasons to be pleased when we have recognized a triangle in the proposed figure. In fact, we know several theorems and have solved various problems about triangles, and one or the other of these known theorems or former solutions may be applicable to our present problem. By recognizing a triangle, we establish contact with an extensive layer of our formerly acquired knowledge, some streak of which might be useful now. Thus, in general, recognizing may lead us to *remembering* something helpful, to mobilization of relevant knowledge.

11.10. Supplementing and regrouping

We have recognized a triangle in the figure and have succeeded in remembering a theorem about triangles that has some chance to be applicable to the present situation. Yet, to actually apply that theorem we must add some auxiliary line to our triangle, for instance, an altitude. Thus, in general, prospective useful elements just mobilized may be added to our conception of the problem to enrich it, to make it fuller, to fill in gaps, to supply its deficiencies, in a word, to *supplement* it.

Supplementing introduces new materials into our conception of the problem and is an important step in its organization. Yet sometimes we can make an important advance in organization without introducing any new material, just by changing the disposition of the elements already present, by conceiving them in new relations, by rearranging or *regrouping* them. By regrouping its elements, we change the "structure" of our problem's conception. Thus regrouping means *restructuring*.[4]

Let us state this consideration more concretely. The decisive step in the solution of a geometric problem may be the introduction of an

[4] *Cf.* Duncker, *loc. cit.* pp. 29–30.

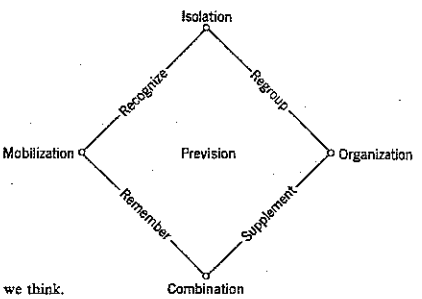


Fig. 11.1. How we think.

others the four vertices, and four more terms are written along the sides.

Mobilization and *organization* are represented by opposite ends of the horizontal diagonal of the square. In fact, these are complementary activities. Mobilization is extracting relevant items from our memory, organization is connecting such items purposefully.

Isolation and *combination* are represented by opposite ends of the vertical diagonal. In fact, these are complementary activities. Isolation is selecting a particular detail from the surrounding whole, combination is assembling dispersed details to a meaningful whole.

The sides adjoining the corner assigned to mobilization are labeled *recognize* and *remember*. In fact, mobilization of items relevant to the problem often starts from recognizing some element given with the problem and consists in remembering connected elements.

The sides adjoining the corner assigned to organization are labeled *supplement* and *regroup*. In fact, organization means supplementing the conception of the problem, making it fuller by adding new details and filling the gaps; and it also means regrouping the whole conception.

As we read the terms along the sides of the square, from left to right, we proceed from mobilized details to the organized whole; a detail just recognized, carefully isolated and focused, may induce a regroupment of the whole conception. Also, a detail remembered which fits into a combination is suitably added to the conception and supplements the whole.

Prevision is the center of our activity aimed at the solution, as the corresponding point is the center of our symbolic square. We keep on mobilizing and organizing, isolating and recombining, recognizing and remembering all sorts of elements, regrouping and supplementing

our conception of the problem, just to foresee the solution, or some feature of the solution, or a bit of the path leading to it. If prevision comes to us abruptly, in a flash, we call it inspiration, or illuminating idea; our central desire is to have such an idea.

The mental operations surveyed in Fig. 11.1 take more specific forms when applied to special material. Thus, correspondingly to the four sides of the square, we list four mental operations important in solving mathematical problems:

Recognize: use definitions	Regroup: transform the problem
Remember: known theorems and problems	Supplement: introduce auxiliary elements

There is another point. The problem solver's moves are prompted or accompanied by feelings of relevancy and proximity, and feelings gauging the goodness of his guess. In discussing this we have mentioned incidentally that more sophisticated people have more differentiated feelings concerning such points. I do not wish to suppress here a rather speculative remark:[5] some such shades of feeling may be connected with the mental operations surveyed in Fig. 11.1.

We cheer up when our conception of the problem appears *well balanced* and *coherent, complete* with all details, and all details are *familiar*. If we have *distinct details* in a *harmonious whole*, the idea of the solution appears *near*. What we express with these terms is, it seems to me, that certain *activities considered above are well progressing*, or have already reached their goal.

Our conception of the problem appears well balanced when we do not feel the need of *regrouping* it, and appears as coherent when we have no trouble in *remembering* its details, but any detail easily recalls the others. When there is no need of *supplementing* it, the conception appears as complete, and it appears as familiar when all details have been *recognized*. Distinctness of details comes from foregoing *isolation* of, and concentration on, each detail, and the harmony of the whole conception results from successful *combination* of the details. We say that the idea is near when we feel that we are well progressing toward fuller *prevision*.

Wishing to arrange these favorable signs of our progress systematically, we place them so that their relative positions are the same as those of the corresponding terms in the square of Fig. 11.1. Thus, we arrange seven terms so as the four sides of that square and the three important points on its vertical diagonal are disposed. See the scheme:

[5] *Cf.* HSI, Signs of progress 4, p. 184.

Well isolated:
distinct details

Well recognized:
familiar

Well grouped:
well balanced

Prevision promising:
idea near

Well remembered:
coherent

Well supplemented:
complete

Well combined:
harmonious whole

11.13. The part suggests the whole

A whistling boy passed me in the street and I caught one or two measures of a melody which I like very much but had not heard for a long time. Suddenly that melody filled by mind, ousting completely whatever worries or idle thoughts I had before.

This little event is a good illustration of the "association of ideas," a phenomenon already described by Aristotle and by many authors after him. Bradley gives a good description: "Any part of a single state of mind tends, if reproduced, to reinstate the remainder." In fact, in my case, one measure brought back the whole impact of that melody and then, by and by, the remaining measures. Here is another description which lacks essential details but is easy to remember: "The part suggests the whole." Let us regard this short sentence as a convenient abbreviation of Bradley's more precise formulation.

Notice the important words "tends" and "suggests." The statements

"The part suggests the whole"

"The part tends to reinstate the whole"

"The part has a chance to reinstate the whole"

may be acceptable, but the sentence

"The part reinstates the whole"

is certainly inacceptable as an expression for the "law of association": there is no necessity of recall, just a chance, a tendency. We also know something about the strength of that tendency; a part more in the focus of attention suggests the whole stronger; several parts jointly suggest the whole stronger than any one of them singly. These additions are important if we wish to understand the role of association in the problem solver's mental experience.

Let us consider a strongly schematized example. A mathematical problem can be quickly solved by the application of a certain decisive theorem D, but it is very difficult to solve it without D. At the outset, the problem solver does not even suspect that the theorem D is relevant to his problem, although he is quite well acquainted with the theorem D

itself. How can he discover the decisive role of D? There are various cases.

The case is relatively simple if the proposed problem and the theorem D have a common component part. The problem solver, after having tried this and that, will come upon that component part, isolate and focus it, and then that common part has a chance to recall or "reinstate" the whole theorem D.

The case is less simple if the original conception of the problem and the decisive theorem D have no common component. Still, if there is another theorem C, also known to the problem solver, that has some component in common with the problem and another component in common with D, the problem solver may attain D by first contacting C and then passing from C to D.

Of course, the chain of associations may be still longer; the proposed problem may be in associative contact with A, A with B, B with C, and finally C with D. The longer the chain the longer must the problem solver "shake the bag" or "shake the sieve" till the decisive D eventually falls out.

Shaking the bag or the sieve is a metaphorical way to describe the problem solver's mental experience (see the quotation prefixed to this chapter). The foregoing sections summarized by Fig. 11.1 attempted to describe this experience somewhat less metaphorically. There is a quite plausible interpretation of the activities described; through them, the problem solver seeks to establish desirable associative contacts.

In fact, in recognizing an element, the problem solver places it in a context with which it has strong associative contact. Any newly mobilized element, added to the problem's conception, offers chances to attract further elements with which it is in associative contact. When the problem solver isolates and focuses an element, the attention spent on it gives it more chance to bring in associated elements. A regroupment may bring together elements which could exercise more associative attraction jointly than anyone could singly.

It is, however, hardly possible to explain the problem solver's mental experience by association *alone;* there must be something else besides associative attraction to distinguish between relevant and irrelevant, desirable and undesirable, useful and useless associated elements and combinations.[6]

Examples and Comments on Chapter 11

11.1. *Your experience, your judgment.* The aim of this book is to improve your working habits. In fact, however, only you yourself can improve your own habits. You should find out the difference between what you are usually doing and what you ought to do. This chapter was written to help you to see better what you are usually doing.

[6] *Cf.* Duncker, *loc. cit.* p. 18.

The following exercises, ex. 11.2–11.6, ask you to illustrate passages of the foregoing text. In the first place, try to find illustrations from your own work—such illustrations as come to your mind spontaneously have the best chance to be illuminating. Try to judge with an open mind whether the descriptions in the text or the illustrations in the solutions agree with your experience.

11.2. *Mobilization.* Recall your work on some problem of geometry where the figure, originally almost empty, became more and more filled by auxiliary elements as the solution progressed.

11.3. *Prevision.* Can you recall a case in which, at a pretty definite moment, you became suddenly convinced that the solution will succeed?

11.4. *More parts suggest the whole stronger.* Can you agree, judging by your own experience?

11.5. *Recognizing.* Can you recall a case in which recognizing an element (noticing its formerly unnoticed familiar role) appeared as the turning point of the solution?

11.6. *Regrouping.* Can you recall a case in which regrouping the figure appeared as the key to the solution?

11.7. *Working from inside, working from outside.* Establishing contacts between the proposed problem and his previous experience is certainly an essential part of the problem solver's performance. He can try to discover such contacts "from inside" or "from outside." He may remain within the problem, examining its elements till he finds one that is capable of attracting some usable element from outside, that is, from his previously acquired knowledge. Or he may go outside the problem, examining his previously acquired knowledge until he finds some element applicable to his problem. Working from inside, the problem solver scans his problem, its component parts, its aspects. Working from outside, he surveys his existing knowledge, and ransacks the provinces of knowledge that are most likely to be applicable to the present problem. The two parts of Fig. 11.2 attempt to give visual expression to "inside" and "outside" work.

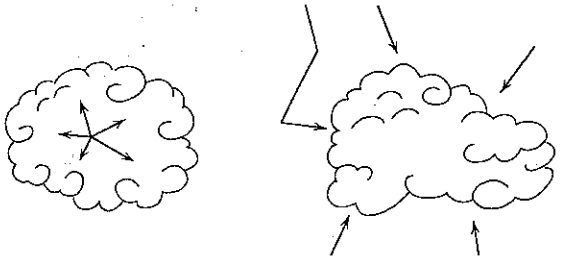


Fig. 11.2. Working from inside, working from outside—to pierce the clouds.