11.2. Having a problem

An essential ingredient of the problem is the desire, the will, and the resolution to solve it. A problem that you are supposed to do and which you have quite well understood, is not yet your problem. It becomes your problem, you really have it, when you decide to do it, when you desire to solve it.

You may be involved more or less deeply in your problem—your desire to solve it may be more or less strong. Unless you have a very strong desire, your chances to solve a really hard problem are not negligible. The desire to solve your problem is a productive desire: it may eventually produce the solution, it certainly produces a change in your mental behavior.

11.3. Relevance

You may have a problem so badly that the problem has you; you cannot get rid of your problem, it follows you everywhere.

A man with a problem may be obsessed by his problem. He appears absentminded; he does not notice things which appear obvious to his neighbors, and he forgets things which none of his neighbors would forget. Now, working intensely on his problems, often forgot to eat his meals.

Yes, the problem solver's attention is selective; it refuses to dwell on things which appear irrelevant to his problem and ignore the most miraculous things that appear relevant. It is a "spying" attention as Leibnitz put it.

11.4. Proximity

A student takes a written examination in mathematics. He is not required to do all the proposed problems, but he should do as many as possible. In this situation he has his strategy may be to start by looking through all the problems at an appropriate pace and choose those he is most likely to master.

Observe that this supposes that this problem solver is able to assess to some extent the difficulty of his problems, that he can estimate to a degree his "psychological distance" from his problem's solution. In fact, anybody seriously concerned with his problem has a vivid feeling for the proximity of the solution and for the pace of his progress toward the solution. He may not use words but he feels keenly: "It goes well, the solution may be just around the corner," or "It goes so slowly and the solution is still far off," or "I got stuck, there is no progress at all," or "I am drifting away from the solution."

11.5. Prevention

As soon as we are seriously concerned with our problem, we try to foresee, we try to guess; we expect something, we anticipate an outline of the shape of the solution. This outline may be more or less definite—and of course it may be more or less wrong, although I would say not very often wrong.

All problem solvers guess, but the sophisticated and the unsophisticated guess somewhat differently.

A primitive person just sits there with his problem, scratching his head or chewing his pencil, waiting for a bright idea, and doing little or nothing to bring that bright idea nearer. And when the desired idea eventually appears and brings a plausible guess, he simply accepts that guess, regarding it as the solution with little or no criticism.

A more sophisticated problem solver takes his guess more skeptically. His guess may be: "There are 25" or "I should tell him this and that." Yet then he checks his guess and may change it: "No, not 25. Yet let me try 30" or "No. It is no use to tell him that, because he could answer thus and so. Yet I could tell him that. . . ." And eventually, by "trial and error," by successive approximations, the problem solver may arrive at the right answer, at an appropriate plan.

A still more sophisticated and more experienced problem solver, when he does not succeed in guessing the whole answer, tries to guess some part of the answer, some feature of the solution, some approach to the solution, or some feature of an approach to the solution. Then he seeks to expand his guess, but also seeks opportunely to check his guess, and to seek to adapt his guess to the best information he can get at the moment.

Of course, both the sophisticated and the unsophisticated would like to have a really good guess, a bright idea. And everybody would like to know what chances his guess has to come true. Such chances cannot be precisely evaluated (this is not the place to discuss remote possibilities of evaluation). Many times, however, the problem solver has a definite feeling about the prospects of his guess. Primitive people who do not even know what a proof is may have the strongest feelings about their guesses; sophisticated people may distinguish fine shades of feeling; but anybody who has conceived a guess has some feeling about the likely fate of his guess. And so we notice still another sort of feeling, besides the feelings of relevance and proximity, in the problem solver's mind.

Is this point relevant? How far off is the solution? How good is this guess? Such questions accompany each move of the problem solver; they are more felt than formulated and the answers, too, are more felt than formulated. Do such feelings guide the problem solver or do they merely accompany his decisions? Are they causes or symptoms? I don't know, but I do know that if you do not have such feelings, you are not really concerned with your problem.

11.6. Region of search

I seldom part with my wrist watch, but when I do I usually have some trouble to find it. When I take my watch, I habitually start looking for it at the well-defined place: on my desk, or on a certain shelf where I always store my and belongings, or wherever I have placed it if I happen to remember that I took off my wrist watch just there.

Such behavior is typical. As soon as we are seriously concerned with our problem, we anticipate an outline of its solution. This outline may be vague, it may be hardly conscious, but it manifests itself in our behavior. We may try various solutions, but they are all alike; they are all within that preconceived, but perhaps not consciously preconceived, outline. When none of the solutions tried fits the problem, we feel lost, nothing else comes to mind; we cannot step outside that preconceived outline. We do not look for just any kind of solution, but for a certain kind, a kind within a limited outline. We do not look for a solution just anywhere in the world, but for a solution within a certain limited region of search.2

To begin our search within a likely limited region may be reasonable. When I am trying to find my missing wrist watch, it is quite reasonable not to look for it anywhere in the universe, or anywhere in the city, or anywhere in the house, but just on my desk where I found it several times in the past. It is quite reasonable to begin by seeking the unknown within that limited region, but it is unreasonable to persevere in seeking it there even when it becomes more and more clear that it is not there.

11.7. Decisions

Problem solving may be contemplative; with primitive people, it may be inarticulate brooding. Or it may be a long, strenuous, winding road to the solution, each turning of which is marked by a decision. Such decisions are prompted (or perhaps merely accompanied) by feelings of relevancy and proximity, by swelling or fading hope. Decisions and prompting feelings are seldom expressed in words, but may be occasionally:

1 Otto Dantzig, "Problem Solving, Psychologische Monographie, vol. 58, No. 5 (1941)." See p. 25.
11.8. Mobilization and organization

The problem solver's mental activity is very imperfectly known and its complexity may be unattainable. Yet one result of this activity is perfectly obvious: as the problem solver advances, he collects more and more material.

Let us compare the problem solver's conception of a mathematical problem at the beginning and at the end of his work. When the problem arises, there is simply a picture: the problem solver sees his problem as an undivided whole without details, or with very few details; for instance, he may see just the principal parts, unknown, data, and conditions, or perhaps the solution and configuration. Solving a problem is very different: it is complex, full of added details and materials the relevancy of which the problem solver could hardly have suspected at the outset. There are auxiliary lines in the originally almost empty geometric figure, there are auxiliary unknowns, there are materials from the formerly acquired knowledge of the problem solver, especially theorems applied to the problem. That just these theorems will be applicable, the problem solver did not foresee at all at the beginning.

Where do all these materials, auxiliary elements, theorems, etc. come from? The problem solver has collected them; he had to extract them from his memory and purposefully connect them with his problem. We call such collecting mobilization and such connecting organization.

Mobilization and organization cannot actually be separated; they are complementary aspects of the same complex process—of our work aimed at the solution. Such work, when extensive, brings into play all our psychical resources, requires the whole gamut of our mental activities, and presents an inexhaustible variety of aspects. We may be tempted to distinguish some of the manifold mental operations involved and describe them by such terms as isolation and combination, recognizing and remembering, regrouping and supplementing.

11.11. Isolation and combination

When we are examining a complex whole, our attention may be attracted now by this detail and then by another. We concentrate on a certain detail, we focus on it, we emphasize it, we single it out, we distinguish it from its surroundings, in one word, we isolate it. Then the spotlight shifts to another detail, we isolate still another detail, and so on.

After examining various details and revealing some of them, we may feel the need of visualizing again the situation as a whole. In fact, after the revaluation of some details, the appearance of the whole, the "vue d'ensemble," the "Gestalt" may have changed. The combined effect of our reassessment of certain details may result in a new mental picture of the whole situation, in a new, more harmonious combination of all the details.

Isolation and combination may advance the solution in complementing each other. Isolation leads to decomposing the whole into its parts, a subsequent combination reassembles the parts into a more or less different whole. Decomposed and recombined, again decomposed and again recombined, our view of the problem may evolve toward a more promising picture.

11.12. A diagram

A diagrammatic summary of the foregoing sections is offered by Fig. 11.1, which the reader should take for what it is worth. Nine terms are arranged in a square; one occupies the center of the square, four others the four vertices, and four more terms are written along the sides.

Mobilization and organization are represented by opposite ends of the horizontal diagonal of the square. In fact, these are complementary activities. Mobilization is extracting relevant items from our memory, organization is connecting such items purposefully.

Isolation and combination are represented by opposite ends of the vertical diagonal. In fact, these are complementary activities. Isolation is selecting a particular detail from the surrounding whole, combination is assembling dispersed details to a meaningful whole.

The sides adjoining the corner assigned to mobilization are labeled recognize and remember. In fact, mobilization of items relevant to the problem often starts from recognizing some element given with the problem and consists in remembering connected elements.

The sides adjoining the corner assigned to organization are labeled supplement and regroup. In fact, organization means supplementing the conception of the problem, making it fuller by adding new details and filling the gaps; and it also means regrouping the whole conception.

As we read the terms along the sides of the square, from left to right, we proceed from mobilized details to the organized whole; a detail just recognized, carefully isolated and focused, may induce a regroupment of the whole conception. Also, a detail remembered which fits into a combination is suitably added to the conception and supplements this whole.

Prevision is the center of our activity aimed at the solution, as the corresponding point is the center of our symbolic square. We keep on mobilizing and organizing, isolating and recombining, recognizing and remembering all sorts of elements, regrouping and supplementing.
our conception of the problem, just to foresee the solution, or some feature of the solution, or a bit of the path leading to it. If preview comes to us abruptly, in a flash, we call it inspiration, or illuminating idea; our central desire is to have such an idea.

The mental operations surveyed in Fig. 11.1 take more specific forms when applied to special material. Thus, correspondingly to the four sides of the square, we list four mental operations important in solving mathematical problems:

**Recognize:**
- Use definitions
- Known theorems and problems

**Regroup:**
- Transform the problem
- Supplement: introduce auxiliary elements

**Remember:**
- Familiar
- Distinct details
- Complete with all details, and all details are familiar

**Well remembered:**
- Coherent
- Harmonious whole
- Balanced

There is another point. The problem solver’s moves are prompted or accompanied by feelings of relevancy and proximity, and feelings gauging the goodness of his guess. In discussing this we have made it clear that more sophisticated people have more differentiated feelings concerning such points. I do not wish to suppress here a rather speculative remark: some such states of feeling may be connected with the mental operations surveyed in Fig. 11.1.

We cheer up when our conception of the problem appears well balanced and coherent, complete with all details, and all details are familiar. If we have distinct details in a harmonious whole, the idea of the solution appears near. What we express with these terms is, it seems to us, that certain activities considered above are well progressing, or we already reached their end.

Our conception of the problem appears well balanced when we do not feel the need of regrouping it, and appears as coherent when we have no trouble in remembering its details, but any detail easily recalls the others. When there is no need of supplementing it, the conception appears as complete, and it appears as familiar when all details have been recognized. Distinctness of details comes from foregoing isolation of, and concentration on, each detail, and the harmony of the whole conception results from successful combination of the details. We say that the idea is near when we feel that we are well progressing toward fuller perception.

Wishing to arrange these favorable signs of our progress systematically, we place them so that their relative positions are the same as those of the corresponding terms in the square of Fig. 11.1. Thus, we arrange seven terms so as the four sides of that square and the three important points on its vertical diagonal are disposed. See the scheme.

\[ \text{Cf. Hilf, Sign of progress 4, p. 184.} \]

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**11.13. The part suggests the whole**

A whistling boy passed me in the street and I caught one or two measures of a melody which I like very much but had not heard for a long time. Suddenly that melody filled my mind, outshining completely whatever worries or idle thoughts I had before.

This little event is a good illustration of the “association of ideas,” a phenomenon already described by Aristotle and by many authors after him. Bradley gives a good description: “Any part of a single state of mind tends, if reproduced, to reinstate the remainder.” In fact, in my case, one measure brought back the whole impact of that melody and then, by and by, the remaining measures. Here is another description which lacks essential details but is easy to remember: “The part suggests the whole.” Let us regard this short sentence as a convenient abbreviation of Bradley’s more precise formulation.

Notice the important words “tends” and “suggests.” The statements “The part suggests the whole” and “The part tends to reinstate the whole” are certainly unacceptable as an expression for the “law of association”; there is no necessity of recall, just a chance, a tendency. We also know something about the strength of that tendency: a part more in the focus of attention suggests the whole stronger; several parts jointly suggest the whole stronger than any one of them singly. These additions are important if we wish to understand the role of association in the problem solver’s mental experience.

Let us consider a strongly schematized example. A mathematical problem can be quickly solved by the application of a certain decisive theorem D, but it is very difficult to solve it without D. At the outset, the problem solver does not even suspect that the theorem D is relevant to his problem, although he is quite well acquainted with the theorem D itself.

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**11.1. Your experience, your judgment**

The aim of this book is to improve your working habits. In fact, however, only you yourself can improve your own habits. You should find out the difference between what you are usually doing and what you ought to do. This chapter was written to help you to see better what you are usually doing.

The following exercises, ex. 11.2–11.6, ask you to illustrate passages of the foregoing text. In the first place, try to find illustrations from your own work: such illustrations are sure to come to your mind spontaneously have the best chance to be illuminating. Try to judge with an open mind whether the descriptions in the text or the illustrations in the solutions agree with your experience.

**11.2. Mobilisation.** Recall your work on some problem of geometry where the figure, originally almost empty, became more and more filled by auxiliary elements as the solution progressed.

**11.3. Precision.** Can you recall a case in which, at a pretty definite moment, you became suddenly convinced that the solution will be quite clear.

**11.4. More parts suggest the whole stronger.** Can you agree, judging by your own experience?

**11.5. Recognising.** Can you recall a case in which recognising an element (noticing its formerly unnoticed familiar role) appeared as the turning point of the solution?

**11.6. Regrouping.** Can you recall a case in which regrouping the figure appeared as the key to the solution?

**11.7. Working from inside, working from outside.** Establishing contacts between the proposed problem and his previous experience is certainly an essential part of the problem solver’s performance. He can try to discover such contacts “from inside” or “from outside.” He may remain within the problem, examining its elements till he finds one that is capable of attracting some useful element from outside, that is, from his previously acquired knowledge. Or he may go outside the problem, examining his previously acquired knowledge until he finds something applicable to his problem. Working from inside, the problem solver seeks his problem, its component parts, its aspects. Working from outside, he surveys his existing knowledge, and on this basis he proceeds to the present problem. The two parts of Fig. 11.2 attempt to give visual expression to “inside” and “outside” working.

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**Fig. 11.2. Working from inside, working from outside—to pierce the cloud.**

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\*Cf. Driker, loc. cit. p. 18.

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