This textbook uses the term “co-domain” where your textbook used the term “target”

1. Let \( X = \{1, 3, 5\} \) and \( Y = \{s, t, u, v\} \). Define \( f: X \to Y \) by the following arrow diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
1 \cdot & \cdot & s \\
3 \cdot & \cdot & t \\
5 \cdot & \cdot & u \\
& \cdot & v \\
\end{array}
\]

a. Write the domain of \( f \) and the co-domain of \( f \).
b. Find \( f(1), f(3), \text{ and } f(5) \).
c. What is the range of \( f \)?
d. Is 3 an inverse image of \( s \)? Is 1 an inverse image of \( u \)?
e. What is the inverse image of \( s \)? Of \( u \)? Of \( v \)?
f. Represent \( f \) as a set of ordered pairs.

2. Let \( X = \{1, 3, 5\} \) and \( Y = \{a, b, c, d\} \). Define \( g: X \to Y \) by the following arrow diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
1 \cdot & \cdot & a \\
3 \cdot & \cdot & b \\
5 \cdot & \cdot & c \\
& \cdot & d \\
\end{array}
\]

a. Write the domain of \( g \) and the co-domain of \( g \).
b. Find \( g(1), g(3), \text{ and } g(5) \).
c. What is the range of \( g \)?
d. Is 3 an inverse image of \( a \)? Is 1 an inverse image of \( b \)?
e. What is the inverse image of \( b \)? Of \( c \)?
f. Represent \( g \) as a set of ordered pairs.

3. Indicate whether the statements in parts (a)–(d) are true or false. Justify your answers.
   a. If two elements in the domain of a function are equal, then their images in the co-domain are equal.
   b. If two elements in the co-domain of a function are equal, then their preimages in the domain are also equal.
   c. A function can have the same output for more than one input.
   d. A function can have the same input for more than one output.

4. a. Find all functions from \( X = \{a, b\} \) to \( Y = \{u, v\} \).
   b. Find all functions from \( X = \{a, b, c\} \) to \( Y = \{u\} \).
   c. Find all functions from \( X = \{a, b, c\} \) to \( Y = \{u, v\} \).

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1. The definition of one-to-one is stated in two ways:
   \[ \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2 \]
   \[ \forall x_1, x_2 \in X, \text{ if } x_1 \neq x_2 \text{ then } F(x_1) \neq F(x_2). \]
   Why are these two statements logically equivalent?

2. Fill in each blank with the word most or least.
   a. A function \( F \) is one-to-one if, and only if, each element in the co-domain of \( F \) is the image of at ______ one element in the domain of \( F \).
   b. A function \( F \) is onto if, and only if, each element in the co-domain of \( F \) is the image of at ______ one element in the domain of \( F \).

3. When asked to state the definition of one-to-one, a student replies, “A function \( f \) is one-to-one if, and only if, every element of \( X \) is sent by \( f \) to exactly one element of \( Y \).” Give a counterexample to show that the student’s reply is incorrect.

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4. Let \( f: X \to Y \) be a function. True or false? A sufficient condition for \( f \) to be one-to-one is that for all elements \( y \) in \( Y \), there is at most one \( x \) in \( X \) with \( f(x) = y \).

5. All but two of the following statements are correct ways to express the fact that a function \( f \) is onto. Find the two that are incorrect.
   a. \( f \) is onto \( \iff \) every element in its co-domain is the image of some element in its domain.
   b. \( f \) is onto \( \iff \) every element in its domain has a corresponding image in its co-domain.
   c. \( f \) is onto \( \iff \forall y \in Y, \exists x \in X \text{ such that } f(x) = y. \)
   d. \( f \) is onto \( \iff \forall x \in X, \exists y \in Y \text{ such that } f(x) = y. \)
   e. \( f \) is onto \( \iff \) the range of \( f \) is the same as the co-domain of \( f \).

6. Let \( X = \{1, 5, 9\} \) and \( Y = \{3, 4, 7\} \).
   a. Define \( f: X \to Y \) by specifying that \( f(1) = 4, f(5) = 7, f(9) = 4 \).
   Is \( f \) one-to-one? Is \( f \) onto? Explain your answers.
b. Define \( g: X \rightarrow Y \) by specifying that
\[
g(1) = 7, \quad g(5) = 3, \quad g(9) = 4.
\]
Is \( g \) one-to-one? Is \( g \) onto? Explain your answers.

7. Let \( X = \{a, b, c, d\} \) and \( Y = \{e, f, g\} \). Define functions \( F \) and \( G \) by the arrow diagrams below.

\[\begin{array}{c|c}
\text{Domain of } F & \text{Co-domain of } F \\
\hline
\bullet a & \bullet f \\
\bullet b & \bullet f \\
\bullet c & \bullet g \\
\bullet d & \bullet g \\
\end{array}\]

\[\begin{array}{c|c}
\text{Domain of } G & \text{Co-domain of } G \\
\hline
\bullet a & \bullet e \\
\bullet b & \bullet f \\
\bullet c & \bullet e \\
\bullet d & \bullet g \\
\end{array}\]

a. Is \( F \) one-to-one? Why or why not? Is it onto? Why or why not?
b. Is \( G \) one-to-one? Why or why not? Is it onto? Why or why not?

8. Let \( X = \{a, b, c\} \) and \( Y = \{w, x, y, z\} \). Define functions \( H \) and \( K \) by the arrow diagrams below.

\[\begin{array}{c|c}
\text{Domain of } H & \text{Co-domain of } H \\
\hline
\bullet a & \bullet w \\
\bullet b & \bullet x \\
\bullet c & \bullet y \\
\end{array}\]

\[\begin{array}{c|c}
\text{Domain of } K & \text{Co-domain of } K \\
\hline
\bullet a & \bullet w \\
\bullet b & \bullet x \\
\bullet c & \bullet y \\
\end{array}\]

a. Is \( H \) one-to-one? Why or why not? Is it onto? Why or why not?
b. Is \( K \) one-to-one? Why or why not? Is it onto? Why or why not?

10. Define \( f: Z \rightarrow Z \) by the rule \( f(n) = 2n \), for all integers \( n \).
   (i) Is \( f \) one-to-one? Prove or give a counterexample.
   (ii) Is \( f \) onto? Prove or give a counterexample.

b. Let \( 2\mathbb{Z} \) denote the set of all even integers. That is, 
\( 2\mathbb{Z} = \{n \in \mathbb{Z} \mid n = 2k \text{ for some integer } k\} \).
Define \( h: Z \rightarrow 2\mathbb{Z} \) by the rule \( h(n) = 2n \), for all integers \( n \).
Is \( h \) onto? Prove or give a counterexample.

11. Define \( g: Z \rightarrow Z \) by the rule \( g(n) = 4n - 5 \), for all integers \( n \).
   (i) Is \( g \) one-to-one? Prove or give a counterexample.
   (ii) Is \( g \) onto? Prove or give a counterexample.

b. Define \( G: \mathbb{R} \rightarrow \mathbb{R} \) by the rule \( G(x) = 4x - 5 \) for all real numbers \( x \).
   Is \( G \) onto? Prove or give a counterexample.
For problems 1-4, find $G \circ F$ and $F \circ G$ and determine whether $G \circ F = F \circ G$

1.

2.

3. $F(x) = x^3$ and $G(x) = x - 1$, for all real numbers $x$.

4. $F(x) = x^5$ and $G(x) = x^{1/5}$ for all real numbers $x$.

5. Define $f: \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = -x$ for all real numbers $x$. Find $(f \circ f)(x)$.

6. Define $F: \mathbb{Z} \to \mathbb{Z}$ and $G: \mathbb{Z} \to \mathbb{Z}$ by the rules $F(a) = 7a$ and $G(a) = a \mod 5$ for all integers $a$. Find $(G \circ F)(0)$, $(G \circ F)(1)$, $(G \circ F)(2)$, $(G \circ F)(3)$, and $(G \circ F)(4)$.

7. Define $H: \mathbb{Z} \to \mathbb{Z}$ and $K: \mathbb{Z} \to \mathbb{Z}$ by the rules $H(a) = 6a$ and $K(a) = a \mod 4$ for all integers $a$. Find $(K \circ H)(0)$, $(K \circ H)(1)$, $(K \circ H)(2)$, and $(K \circ H)(3)$.

8. Define $L: \mathbb{Z} \to \mathbb{Z}$ and $M: \mathbb{Z} \to \mathbb{Z}$ by the rules $L(a) = a^2$ and $M(a) = a \mod 5$ for all integers $a$.
   a. Find $(L \circ M)(12)$, $(M \circ L)(12)$, $(L \circ M)(9)$, and $(M \circ L)(9)$.
   b. Is $L \circ M = M \circ L$?